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AUTHORS' CLOSURE TO DISCUSSION[1] BY M. J. LOWREY

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The authors thank Dr. Lowrey for his discussion, and for pointing out the numerical error in Fig. 2[1]. They agree with that the results obtained by the lumped mass conventional finite strip (matrix) method and the lumped mass finite strip-difference calculus technique are the same.

It is well known that the convergence of the consistent mass approximation is faster than the lumped mass approximation. The authors do not agree with the discusser's statement that in the conventional finite strip method the increased accuracy (of the consistent mass approximation) may be achieved with little increase in computation time (over the lumped mass approximation). Since the distributed mass, lumped as concentrated line masses at the strip interfaces, exerts only lateral inertial forces and no inertial moments during free vibration, only the lateral deflections at the interfaces, w_i, contribute to the dynamic degrees-of-freedom of the system. The rotations at the interfaces (θ_i) are dependent on these independent degrees-of-freedom. Hence, under the lumped mass approximation, we essentially solve a $(N_s - 1)$† degrees-of-freedom system, where N_s is the number of strips. In the consistent mass approximation, the rotations also contribute to the dynamic degrees-of-freedom, and we solve a $(2N_s)$‡ degrees-of-freedom system. The mathematical reduction of the eigen-analysis of the $(2N_s)$th order matrix to the eigen-analysis of the $(N_s - 1)$th order matrix is given in Appendix A. Since the order of the matrix involved is reduced, the computation time for the lumped mass approximation is considerably less than that for the consistent mass approximation.

Since the authors' finite strip–difference calculus technique presented in the paper was based on the lumped mass approximation, it was compared with the conventional lumped mass finite strip method. Recently the authors have extended the technique for the consistent mass approximation also (Appendix B). The results given in Table 1 are practically identical with the conventional consistent mass finite strip method results given by discusser.

Summing up, the conventional finite strip (matrix) method and the authors' finite strip–difference calculus technique give identical results, for both the lumped mass and the

Table 1. Consistent mass finite strip–difference calculus results for the example plate

N_s	ω_{11}^2	ω_{12}^2	ω_{13}^2	ω_{14}^2	ω_{15}^2
2	152·266				
3	152·213	392·257			
4	152·205	390·477	1048·35		
5	152·203	389·980	1037·17	2511·07	
6	152·202	389·802	1032·95	2473·59	5331·04
8	152·202	389·688	1030·19	2447·84	5192·42
10	152·202	389·658	1029·42	2440·48	5150·62
20	152·202	389·638	1028·92	2435·56	5122·05
Exact	152·202	389·636	1028·88	2435·23	5120·07

†The lateral deflections at the interfaces $1, 2, 3 \ldots (N_s - 1)$. (The deflections at 0 and N_s are zero because of the simply supported boundary conditions).

‡$(N_s - 1)$ lateral deflections plus the rotations at the interfaces $0, 1, 2 \ldots N_s$.

consistent mass approximations. The advantages of the authors' method are: (a) It does not involve any matrix operations, and requires far less computational work, (b) the computational work involved is independent of the number of strips, and (c) the analytical form of the solution is best suited for parametric studies on the convergence and accuracy of the finite strip approach.

REFERENCE

1. Sundararajan and D. V. Reddy, Finite strip–difference calculus technique for plate vibration problems. *Int. J. Solids Structures* **11**, 425 (1975).

APPENDIX A

The equations of motion of the finite strip assemblage are (from eqn (24) of [1]).

$$([K]-\omega^2[M])\{\delta\}=0 \tag{1}$$

where $\{\delta\}$ is the displacement vector of order $(2N_s)$, and $[K]$ and $[M]$ are stiffness and mass matrices of the order $(2N_s)$. The matrix eqn (1) can be partitioned into

$$\left[\begin{bmatrix} K_{ww}K_{w\theta} \\ K_{\theta w}K_{\theta\theta}\end{bmatrix}-\omega^2\begin{bmatrix} M_{ww}M_{w\theta} \\ M_{\theta w}M_{\theta\theta}\end{bmatrix}\right]\begin{Bmatrix}\delta_w \\ \delta_\theta\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \tag{2}$$

where ω is the frequency of vibration, $\{\delta_w\}$ is a (N_s-1)th order vector containing the lateral deflections at the strip interfaces $1,2,3,\ldots(N_s-1)$, and δ_θ is a (N_s+1)th order vector containing the rotations at the strip interfaces $0,1,2,\ldots N_s$. (The lateral deflections at the interfaces 0 and N_s are zero because of the simply supported boundary conditions). K_{ww}, $K_{w\theta}$, ... etc. are submatrices of the appropriate order. Because of the nature of the lumped mass matrix (eqn (20) of [1]), eqn (2) reduces to,

$$\left[\begin{bmatrix} K_{ww}K_{w\theta} \\ K_{\theta w}K_{\theta\theta}\end{bmatrix}-\frac{\mu d}{2}\omega^2\begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}\right]\begin{Bmatrix}\delta_w \\ \delta_\theta\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \tag{3}$$

where μ = mass per unit area, d = width of strip and I is a unit matrix. Equation (3) can be written as two matrix equations,

$$\left([K_{ww}]-\frac{\mu d}{2}\omega^2[I]\right)\{\delta_w\}+[K_{w\theta}]\{\delta_\theta\}=\{0\} \tag{4a}$$

$$([K_{\theta w}]\{\delta_w\}+[K_{\theta\theta}])\{\delta_\theta\}=\{0\} \tag{4b}$$

Substituting eqn (4b) in eqn (4a) gives

$$\left([K^*]-\frac{\mu d}{2}\omega^2[I]\right)\{\delta_w\}=\{0\} \tag{5}$$

where

$$[K^*]=[K_{ww}]-[K_{w\theta}][K_{\theta\theta}]^{-1}[K_{\theta w}]. \tag{6}$$

So, the frequency parameters $[(\mu d/2)\omega^2]$ are the eigenvalues of the (N_s-1)th order square matrix $[K^*]$.

In the consistent mass approximation, where the mass matrix cannot be partitioned into unit and null matrices as in eqn (3), the frequency parameters ω^2 are obtained from eqn (1) as the eigenvalues of the $(2N_s)$th order square matrix $[M]^{-1}[K]$.

APPENDIX B

The consistent mass matrix for a typical strip, based on the deflection functions defined by eqns (1) and (2) of [1] is:

$$[m]=\begin{bmatrix} \frac{13\mu d}{35} & & \text{Symmetric} & \\ \frac{11\mu d^2}{210} & \frac{\mu d^3}{105} & & \\ \frac{9\mu d}{70} & \frac{13\mu d^2}{420} & \frac{13\mu d}{35} & \\ \frac{-13\mu d^2}{420} & \frac{-3\mu d^3}{420} & \frac{-11\mu d^2}{210} & \frac{\mu d^3}{105}\end{bmatrix} \tag{7}$$

Similar to eqn (31) of [1], the mass coefficients also satisfy the relationships,

$$\begin{aligned} m_{11}&=m_{33}\\ m_{12}&=m_{21}=-m_{34}=-m_{43}\\ m_{13}&=m_{31}\\ m_{14}&=m_{41}=-m_{23}=-m_{32}\\ m_{22}&=m_{44}\\ m_{24}&=m_{42}\end{aligned} \tag{8}$$

Substituting eqns (7) and (8) in eqns (30a) and (30b) of [1], gives

$$[(k_{13}-\omega^2 m_{13})(E^{-1}+E)+2(k_{11}-\omega^2 m_{11})]w_j+[(k_{14}-\omega^2 m_{14})(E-E^{-1})]\theta_j=0 \tag{9a}$$

and

$$[(k_{14}-\omega^2 m_{14})(E^{-1}-E)]w_j+[(k_{24}-\omega^2 m_{24})(E+E^{-1})+2(k_{22}-\omega^2 m_{22})]\theta_j=0. \tag{9b}$$

Substituting eqn (9b) into eqn (9a)

$$(k_{13}-\omega^2 m_{13})(E^{-1}+E)+2(k_{11}-\omega^2 m_{11})+\frac{[(k_{14}-\omega^2 m_{14})^2(E^{-1}-E)^2]}{[(k_{24}-\omega^2 m_{24})(E+E^{-1})+2(k_{22}-\omega^2 m_{22})]}w_j=0. \tag{10}$$

For a plate, simply supported at $x=0(j=0)$ and $x=1(j=N_s)$, w_j may be assumed to be

$$w_j=\sum_{n=1,2} A_n \sin\frac{n\pi j}{N_s}. \tag{11}$$

Substituting eqn (11) in eqn (10) gives

$$\left[k_{13}\cos\frac{n\pi}{N_s}+k_{11}-\omega^2 m_{13}\cos\frac{n\pi}{N_s}-m_{11}\omega^2\right]-\frac{\left[k_{14}^2\sin^2\frac{n\pi}{N_S}+\omega^4 m_{14}^2\sin^2\frac{n\pi}{N_S}-2\omega^2 k_{14}m_{14}\sin^2\frac{n\pi}{N_S}\right]}{\left[k_{24}\cos\frac{n\pi}{N_S}+k_{22}-\omega^2 m_{24}\cos\frac{n\pi}{N_S}-\omega^2 m_{22}\right]} \tag{12}$$

$=0$ for $n=1,2,\ldots$

Equation (12) can be rewritten as a quadratic equation in ω^2,

$$A\omega^4+B\omega^2+C=0 \tag{13}$$

where,

$$A=\left[\left(m_{13}\cos\frac{n\pi}{N_s}+m_{11}\right)\left(m_{24}\cos\frac{n\pi}{N_s}+m_{22}\right)-\left(m_{14}^2\sin^2\frac{n\pi}{N_s}\right)\right]$$

$$B=\left[\left(2k_{14}m_{14}\sin^2\frac{n\pi}{N_s}\right)-\left(k_{24}\cos\frac{n\pi}{N_s}+k_{22}\right)\left(m_{13}\cos\frac{n\pi}{N_s}+m_{11}\right)\right.$$
$$\left.-\left(k_{13}\cos\frac{n\pi}{N_s}+k_{11}\right)\left(m_{24}\cos\frac{n\pi}{N_s}+m_{22}\right)\right]$$

$$C=\left[\left(k_{13}\cos\frac{n\pi}{N_s}+k_{11}\right)\left(k_{24}\cos\frac{n\pi}{N_s}+k_{22}\right)-\left(k_{14}^2\sin^2\frac{n\pi}{N_s}\right)\right]. \tag{14}$$

Solution of eqn (13) gives two roots for ω^2. Numerical calculations show that the lower root corresponds to the required ω^2. The example plate is analysed using the present method, and the results are presented in Table 1. They are practically identical with the conventional consistent mass finite strip results given by the discusser.